MORE MEANINGFUL INSIGHTS ON PORTFOLIO AND VALUE AT RISK, AT A BASIC LEVEL

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Abstracts

This paper analyzes the basic connotation of some concepts, proposition, lemma and theorems in financial mathematics, and an order for these contents established to give more meaningful insights on portfolio and value at risk at a basic level. Some problems are introduced as examples to elucidate the concepts and theorems via the detailed solutions.

Keywords*: Finance Mathematics, Portfolio, Value-at-Risk, No-Arbitrage, Self-financing strategy.*

Hiểu biết đầy đủ hơn về Danh mục đầu tư và Giá trị rủi ro, ở mức cơ bản

Tóm tắt

Bài báo này trình bày và phân tích các nội hàm cơ bản của một số khái niệm, bổ đề, định đề và định lý trong toán tài chính theo một trật tự được thiết lập để nhằm đem lại hiểu biết đầy đủ hơn về danh mục đầu tư và giá trị rủi ro của danh mục đầu tư. Một số bài toán đơn giản được giới thiệu như là ví dụ để làm sáng tỏ các nội hàm nhờ lời giải chi tiết dễ hiểu được thực hiện.

Từ khóa: *Toán tài chính, danh mục đầu tư, giá trị rủi ro.*

1. Introduction

The trade process is a stochastic process of transactions which were impacted by a number of exogenous factors with their potential impact levels have been being very difficult to predict unambiguously. The realizations of this process, such as periodicity, up-and-down size, transaction value and others, are a source of information to market participants. They cause prices to move, into full swing some time and the stock price either hit the ceiling or fell the floor. They affect the market maker's beliefs about the value of the stock. For each period the finance market investors may draw inferences about the future value of an asset from its trading history. A preponderance of buy orders may signal good news, causing traders to raise their expected value for the asset. A preponderance of sell orders may induce the opposite revision. At a basic level related to portfolio and value at risk, meaningful insights of the theoretical microstructure literature is the first step to draw lessons from the pass and to know the direction of trades. These basics will give investors to make trading strategies and to translate0 goals and constraints of asset management into dynamic, intertemporal, and coherent portfolio decisions. Under special assumptions, myopic portfolio policies are shown to be optimal and constant over time. In general, however, both optimal theoretical portfolios and current portfolio positions are subject to random movements so that periodic monitoring and rebalancing are necessary. With that calculate value at risk allows to optimize worth weights distributed on assets as to get a maximum return with a minimum standard deviation of the invested portfolio. Portfolio goals and strategy selection are important decisions which based on information in the financial market. First information has been treated to estimate the return direction and magnitude, and value at risk for each asset. Then portfolio goals and strategy will be defined. These mentions are an important research topic in the economics literatures. Information in the financial market is generally imperfect, sometime fuzzy or disturbing market. Xia et al. [1] examined the effect of uncertainty about the stock return predictability of optimal dynamic portfolio choice. Peijnenburg [2] studied on ambiguity and the parameters in life-cycle asset allocation. With assuming the volatility of stock return is stochastic, Andrei and Hasler [3] created a dynamic portfolio choice model to investigate how much attention an investor should pay to the news. Obviously from an economics point of view, the unattractive and fuzzy features of information are their inability to separate the concept of risk aversion from the concept of the elasticity of intertemporal substitution.

This paper provides an important mathematical foundation at a basic level to regard information how to get an optimal0 portfolio with its value at risk [4].

2. The portfolio problem in finance mat0hematics

We consider a financial market with $1 + \boldsymbol{d}$ assets on some probability space $(\Omega, \mathcal{F}, P)$, where the assets are priced at two times, at $t = 0$ (today) and $t = 1$ (one period or one year after). All asset prices today are known and given by usually positive constants $S_0^0, S_0^1, \ldots, S_0^d \in \mathbf{R}$. Asset prices in one period, however, are usually unknown today. In most financial markets, not all asset prices in one period are unknown. In practice, there is a riskless asset, often called bank account, which will pay a sure amount in one year. We now assume that S^0 is a bank account and satisfies

$$S_0^0 = 1 \text{and } S_1^0 = 1 + r \qquad (2.1)$$

Where $r > -1$ denotes the interest rate.

We also assume that all asset prices are positive today $S_0^0, S_0^1, \ldots, S_0^d$ and have finite second moments, i.e.

$$\boldsymbol{E}\left[\left(S_t^i\right)^2\right] < \infty, i \in \{0,1,\ldots,d\} \text{ for } t \in 0\{0,1\} \qquad (2.2)$$

To distinguish the riskless asset S^0 and the risky assets $S^1, \ldots, S^d$ we will use the

$S_t = (S_t^1, \ldots, S_t^d)$ and $\overline{S} = (S_t^0, S_t)$ for t $\{0,1\}$ (2.3)

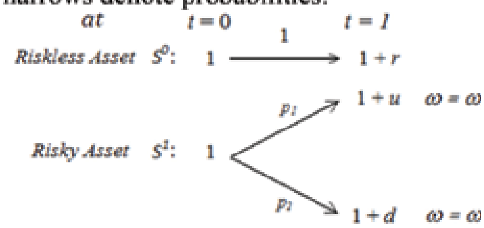
Example 2.1. Let a period binomi model. We assume that $\boldsymbol{d} = 1$, i.e., there only one risky asset, and there are two stat ω_1 and ω_2 of the event world Ω at time t 1, i.e., $\Omega = \{\omega_1, \omega_2\}$. With assuming $S_0^1 =$ we can write

$$S_1^1(\omega_1) = 1 + u \text{ and } S_1^1(\omega_2) = 1 + \qquad (2.4)$$

Where $u > d > -1$. Here u and d a mnemonics for up and down, and it is oft assumed that $u > 0$. The probabilities for and down are given by

$$P(\{\omega_1\}) = p_1 \text{ and } P(\{\omega_2\}) = p_2$$

Where $p_1, p_2 \in (0,1)$ and $p_1 + p_2$ 1. This model can be illustrated as t following diagram, where the numbers on t narrows denote probabilities:



2.1. Trading strategies and arbitra opportunities in one-period markets

Let $\bar{S} = (S_t^0, S_t)$ for $t \in \{0,1\}$ be financial market as above, a trading strateg often called a portfolio is a vector

$$\bar{\vartheta} = \{\vartheta^0, \vartheta\} = \{\vartheta^0, \vartheta^1, \ldots, \vartheta^d\} \in \mathbf{R}^{1+d} \qquad (2.5)$$

Where ϑ^i denotes the number shares held in asset i. At time $t = 0$, the pri today for buying the tradi strategy/portfolio $\bar{\vartheta}$ is (note that $S_0^0 = 1$)

$$\bar{\vartheta}.\bar{S}_0 = \sum_{i=0}^{d} \vartheta^i.S_0^i = (\vartheta^0.1) + (\vartheta.S_0)$$

$$= \vartheta^0 + \vartheta.S_0 \qquad (2.6)$$

At time $t = 1$, i.e., in one year t value of the trading strategy/portfolio

$$\bar{\vartheta}.\bar{S}_1(\omega) = \sum_{i=0}^{d} \vartheta^i.S_1^i(\omega)$$

$$= \vartheta^0(1 + r) + \vartheta.S_1(\omega) \qquad (2.$$

Depending on the state of $\omega \in \Omega$, i. state "up" or "down" in example 1 above,

Definition 2.1. A portfolio trading strategy $\bar{\vartheta} \in \boldsymbol{R}^{1+d}$ is called an arbitrage opportunity for $\bar{S}$ if
$\bar{\vartheta}.\bar{S}_0 \leq 0,\ \bar{\vartheta}.\bar{S}_1 \geq 0\ P_{-a.s.}\ and\ P(\bar{\vartheta}.\bar{S}_1 > 0) > 0$ (2.8)

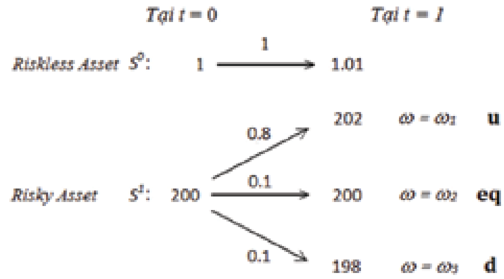
The financial market $\bar{S}$ is called arbitrage-free or no-arbitrage if there is no arbitrage opportunities. In this case one usually says that $\bar{S}$ satisfies **NA**.

Here note that in well-functioning financial markets, arbitrage opportunities do not exist for long time. An investor exploited and sold some assets to make an initial wealth W_0 to buy ϑ^0 riskless asset and ϑ risky assets. If the financial market is frictionless then $\bar{\vartheta}.\bar{S}_0 = W_0$, the value of purchased assets equal to the value of an initial wealth. As some fees such as taxes, brokerage fees, etc., in real financial markets the value of $\bar{\vartheta}.\bar{S}_0 < W_0$. And W_0 is taken as a coordinate origin for referencing, then $\bar{\vartheta}.\bar{S}_0 \leq 0$ are defined. At time $t = 1$, an arbitrage opportunity is exploited, then investor will sell $\bar{\vartheta}$ assets with $\bar{\vartheta}.\bar{S}_1 > 0\ P_{-a.s.}$.

Example 2.2. An arbitrage opportunity.

We consider a financial market $\bar{S} = (S_t^0, S_t)_{t\in\{0,1\}}$, which allows arbitrage transactions. In practice the return of riskless assets is often lower than the one of risky asset. One can, however, salles some high risky assets to buy amount of riskless asset as an approach to hedging. From economics perspective, it should be done when interest rate is higher than inflation rate.

In the following diagram, one financial market is illustrated, where numbers on arrows denote probabilities:



Since existence of "up" states of $\omega \in \Omega$, we can say that the financial market $\bar{S} = (S_t^0, S_t)_{t\in\{0,1\}}$
admits arbitrage. Indeed, we consider a self-financing trade as follows: at time of $t = 0$, a risky asset S_0^1 has been shorted from the market to buy 200 units of riskless asset, i.e., $\vartheta^1 = -1$ (the sign "$-$" denotes one risky asset shorted from the market) and $\vartheta^0 = 200$. It is also inferred that the price of a risky asset is 200 and the price of a riskless asset is 1. Mathematically we set

$$\bar{\vartheta} = (\vartheta^0, \vartheta^1) = (200, -1)$$

Then at time $t = 0$,

$$\bar{\vartheta}.\bar{S}_0 = (\vartheta_0^0, \vartheta_0^1).(S_0^0, S_0^1)$$
$$= (200 \text{ x } 1) + (-1 \text{ x } 200) = 0$$

At $t = 1$,

$$\bar{\vartheta}_1.\bar{S}_1 = (\vartheta_1^0.S_1^0 + \vartheta_1^1.S_1^1)$$
$$= \begin{cases} 200 \text{ x } 1.01 + (-1).202 = 0 & if\ \ \omega = \omega_1 \\ 200 \text{ x } 1.01 + (-1).200 = 2 > 0 & if\ \ \omega = \omega_2 \\ 200 \text{ x } 1.01 + (-1).198 = 4 > 0 & if\ \ \omega = \omega_3 \end{cases}$$

Thus $P(\bar{\vartheta}_1.\bar{S}_1 \geq 0) = 1$ and $P(\bar{\vartheta}_1.\bar{S}_1 > 0) = 0.2$.

Whence $\bar{\vartheta}$ is an arbitrage opportunity.

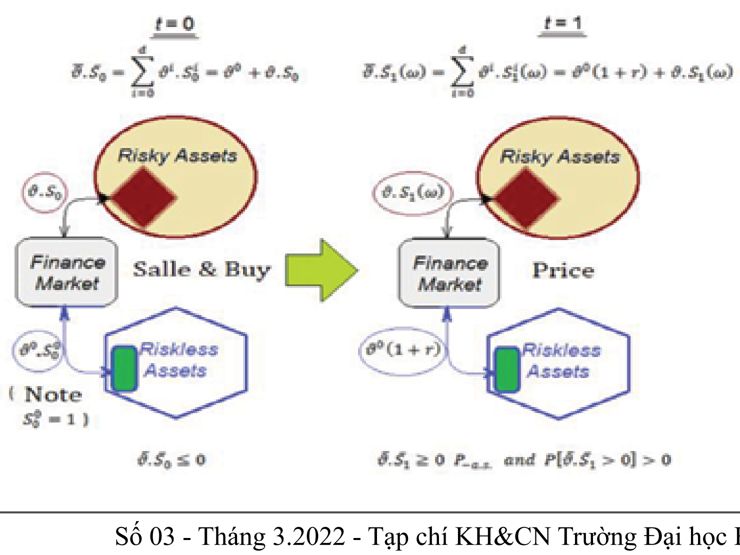


Figure 2.1. Illustration of transactions in the financial market, where at time $t = 0$, some stocks ϑ were shorted from the market whit the price of S_0 to purchase ϑ^0 riskless assets with the price $S_0^0 = 1$. And to period end, at time $t = 1$, $\bar{\vartheta}.\bar{S}_1 \geq 0$ may be obtained.

Remark 1. The remark not only shows existence of arbitrage opportunities (arbitrage portfolios) in financial markets, but indicate us how to make a feasible portfolio, as well as how to get transformed portfolios (portfolio diversification) with updated information of predictable prices of some risky assets. It is most important how to exploit arbitrage opportunities.

If the market $\bar{S}$ admits arbitrage, there is always exists an arbitrage opportunity which satisfies $\bar{\vartheta}.\bar{S}_0 = 0$. Indeed if $\bar{\eta} = (\eta^0, \eta)$ is an arbitrage opportunity with $\bar{\eta}.\bar{S}_0 < 0$.

Set $\bar{\vartheta} := (\vartheta^0, \vartheta) = (\eta^0 - \bar{\eta}.\bar{S}_0, \eta)$ with assumption $S_0^0 = 1$, then

$$\bar{\vartheta}.\bar{S}_0 = \vartheta^0 + \vartheta.S_0 = \eta^0 - \bar{\eta}.\bar{S}_0 + \eta.S_0$$
$$= \eta^0 - (\eta^0.1 + \eta.S_0) + \eta.S_0 = 0$$

Moreover, as $-\bar{\eta}.\bar{S}_0 > 0$,

$$\bar{\vartheta}.\bar{S}_1 = \vartheta^0(1+r).1 + \vartheta.S_1$$
$$= (\eta^0 - \bar{\eta}.\bar{S}_0)(1+r) + \eta.S_1$$
$$= \eta^0(1+r) + (-\bar{\eta}.\bar{S}_0)(1+r) + \eta.S_1$$
$$= [\eta^0(1+r) + \eta.S_1] + (-\bar{\eta}.\bar{S}_0)(1+r)$$
$$= \bar{\eta}.\bar{S}_1 + (-\bar{\eta}.\bar{S}_0)(1+r)$$
$$\geq (-\bar{\eta}.\bar{S}_0)(1+r)$$
$$> 0, \ P_{-a.s.}$$

Thus, it follows from definition 1 that $\bar{\vartheta}$ is an arbitrage opportunity with $\bar{\vartheta}.\bar{S}_0 = 0$.

2.2. Arbitrage and trading in multiperiod discrete-time markets

Consider a financial market on time horizon T, where trade transactions are carried out at different points in time, i.e., at $t = 0, 1, \ldots, T$ with $1 + \boldsymbol{d}$ assets. We also assume that the price of each risky asset is a stochastic process in multiperiod discrete-time market. It follows from that at each point $t = 1, 2, \ldots, T$ the set of asset prices are a random variable and denoted as $S_t = (S_t^1, \ldots, S_t^d)$. For the example 2 above, at $t = 1$ the price of S_1^1 just hit one of three values consisting of 202, 200 and 198 corresponding to the event world $\Omega = \{\omega_1, \omega_2, \omega_3\}$.

As complexity of a multiperiod discrete-time market, first we consider the two-period market with one risky asset.

Example 2.3. Let $(S_t^i)_{t\in\{0,1,2\}}$ be a two-period financial market with a risky asset having the price S_t^i. The price is a random variable. At time $t = 0$, its price is known as S_0^i and at $t = 1$ the price will either be 1 or will be -1. However, at $t = 2$ it could be received one of three values $\{-1,0,2\}$. Then Ω, the world of events possesses all available states $\{-1,0,1,2\}$ which the price S_t^i will be reached, i.e., $\Omega = \{\omega_1 = -1, \omega_2 = 0, \omega_3 = 1, \omega_4 = 2\}$. The following diagram illustrates available paths on which the price could be reached at points in time $t = 1$ and $t = 2$.

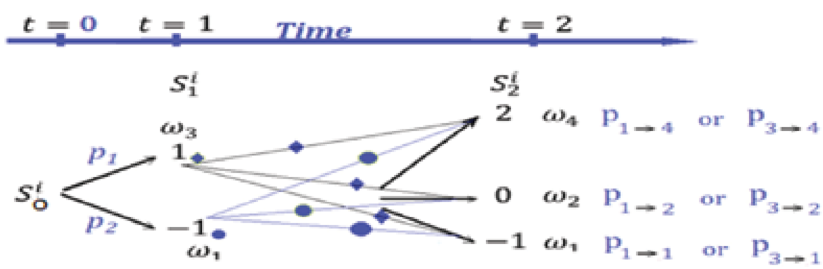


Figure 2.2. Illustration of the two-period market with one risky asset, where its price, S_t^i, is a random variable which could be reached to one of available states of the world $\Omega = \{\omega_1 = -1, \omega_2 = 0, \omega_3 = 1, \omega_4 = 2\}$ with some probability corresponding to every part of each pathway.

As seen in figure 2, at $t = 1$, the price S_1^i just could be either $S_1^i = -1$ or $S_1^i = 1$ corresponding to ω_1 or ω_3. And at $t = 2$ the price S_2^i just also could be received one of three values $\{-1,0,2\}$ corresponding to $\omega_1, \omega_2, \omega_4$, respectively, with the corresponding probabilities $p_{k \to j}$ denoted for the state transition from ω_k to ω_j. And there exist six pathways for the transitions from states at $t = 1$ to states at $t = 2$.

In the case of the multiperiod discrete-time market with a risky asset for T periods, i.e., $t = 1, \dots, T$, the number of the world, N_Ω, is defined by:

$$N_\Omega = \cup_{i=1}^{T} (N_i)$$

where N_i is set of states of the i^{th} period, i.e., where the number of available values the asset price can be reached.

The number of the pathways throughout all periods is

$$N_{Paths} = \Pi_{i=1}^{T} (N_i)$$

Now consider a financial market with the number of risky assets, there exist three ways to represent a portfolio which is made up of these assets: the portfolio based on the proportion of each asset, the portfolio with the number of each asset and the portfolio with the value invested in each asset. We start the portfolio based on the proportion of each asset, because it is easier to understand. Then the other ones will be represented on equivalently.

A self-financing portfolio or from here called simply as a portfolio, is represented by the sequence of vectors $\bar{\eta}_t = (\eta_t^0, \eta_t^1, \dots, \eta_t^d)^T$, and $\sum_{i=0}^{d} \eta_t^i = 1$ for $t = (0,1, \dots, T-1)$. Here $\eta_t^i = \frac{W_t^i}{W_t}$ is the ratio of the value proportion of portfolio W_t^i invested in asset i over the total wealth W_t invested in the portfolio at time t. Equivalently $W_t^i = \vartheta_t^i . S_t^i$ where ϑ_t^i is the number of shares of asset i in the portfolio and S_t^i is the price of asset i, at time t. More precisely

$$W_t = \sum_{i=0}^{d} W_t^i = \sum_{i=0}^{d} \vartheta_t^i . S_t^i = \sum_{i=0}^{d} \eta_t^i . W_t \qquad (2.9)$$

where $\eta_t^i, \vartheta_t^i, W_t$ can be positive, negative or zero, representing long, short or no investment positions in asset i, respectively. In addition, a portfolio strategy can be a sequence of random vectors $\bar{\eta}_t = \{\eta_t^i\}$ that depend on the information obtained from the asset i price before or at time t. In the other word η_t^i is function of random variables S_t^i for $i = (1, \dots, \boldsymbol{d})$ and $t = (1, \dots, T)$. More rigorously we give the definition as following.

Definition 2.2. A self-financing portfolio strategy is given by vector functions $\bar{\eta}_t = (\eta_t^0, \eta_t^1, \dots, \eta_t^d)^T$ such that $\eta_0 \in \mathbf{R}^d$ is a real vector, and for $t = (1, \dots, T-1)$ $\bar{\eta}_t = (\eta_t^0, \eta_t^1, \dots, \eta_t^d)^T$ is a function that maps

$$S_t := \begin{bmatrix} S_0^0 & \cdots & S_T^0 \\ \vdots & \ddots & \vdots \\ S_0^d & \cdots & S_T^d \end{bmatrix} \qquad (2.10)$$

into a vector in $\mathbf{R}^d$ that satisfies $\sum_{i=0}^{d} \eta_t^i = 1$ for $t = (0,1, \dots, T-1)$.

In other words, at time t the function η_t^i depends only the price of all assets from $t = 0$ until time t, not depends on the future prices at points $t + 1$ to T. This shows that a portfolio strategy only depends on the information collected up to the present time, including specific information and forecast information. And a portfolio strategy adjustment can be made out to naturalize risks and could gain more returns.

For more easier in other applications, another representation of the portfolio strategy was effectively confirmed. For example, in the portfolio strategy, it is easier to write portfolio in terms of the ratio above, meanwhile for replicating portfolio of an option, the number of shares of each asset give more convenient representation. In more details, if $\bar{\vartheta}_t = (\vartheta_t^0, \vartheta_t^1, \dots, \vartheta_t^d)$ is known at time t, the value of the portfolio is given by

$$W_t = \bar{\vartheta}_t . \bar{S}_t = \vartheta_t^0 . S_t^0 + \vartheta_t^1 . S_t^1 + \cdots + \vartheta_t^d . S_t^d = \sum_{i=0}^{d} \vartheta_t^i . S_t^i \qquad (2.11)$$

Then the value invested in asset i changes by $\Delta_{W_t^i} = \vartheta_t^i . (S_{t+1}^i - S_t^i)$ from time t to time $t + 1$. And the total value of the portfolio at time $t + 1$ is given by

$$W_{t+1} = \sum_{d=0}^{d} \vartheta_t^i . S_{t+1}^i$$

From time t to time $t+1$, the time-consistent portfolio value is changed and given by

$$W_{t+1} - W_t = \sum_{d=0}^{d} \vartheta_t^i . (S_{t+1}^i - S_t^i)$$

for $t = (0,1,\dots,T-1)$ (2.12)

Here we should also detail the proportion value η_t^i in line of time, i.e., at time $t=0$, $\eta_0^i = \frac{W_0^i}{W_0} = \frac{\vartheta_0^i . S_0^i}{\sum_{i=0}^d W_0^i}$ and at time $t=1$, due to the number of share of each asset is still ϑ_0^i, so $\eta_1^i = \frac{W_1^i}{W_1} = \frac{\vartheta_0^i . S_1^i}{\sum_{i=0}^d W_1^i}$. Then if there exists the change in the number of shares of some assets from ϑ_0^i to ϑ_1^i which will be assigned to second period from time $t=1$ to time $t=2$ where the portfolio is adjusted. Hence the value proportion of portfolio invested in asset i will be adjusted how to satisfy

$$\sum_{i=0}^{d} \eta_t^i = 1 \quad \text{for } t = 0,1,\dots,T-1 \qquad (2.13)$$

This process could be continued and called as rebalancing as described follows

Time	Number of Shares/Units	Value of the Portfolio
At $t = t$ (After Rebalancing at time $t-1$)	$\vartheta_t^i = \frac{\eta_t^i . W_t}{S_t^i}$	$W_t = \vartheta_t^0 S_t^0 + \vartheta_t^1 S_t^1 + \cdots + \vartheta_t^d S_t^d$
$t+1$ Before Rebalancing	$\vartheta_t^i = \frac{\eta_t^i . W_{t+1}}{S_{t+1}^i}$	$W_{t+1} = \vartheta_t^0 S_{t+1}^0 + \vartheta_t^1 S_{t+1}^1 + \cdots + \vartheta_t^d S_{t+1}^d$
$t+1$ After Rebalancing	$\vartheta_{t+1}^i = \frac{\eta_{t+1}^i . W_{t+1}}{S_{t+1}^i}$	$W_{t+1} = \vartheta_{t+1}^0 S_{t+1}^0 + \vartheta_{t+1}^1 S_{t+1}^1 + \cdots + \vartheta_{t+1}^d S_{t+1}^d$

Table 1. Rebalancing the portfolio strategy for $(t+1)^{th}$ period from time t to time $t+1$.

Remarks 2.

- Decisions relating to portfolio rebalancing can be considered an active investment strategy by investors when important information of available prices of some assets in the future for portfolio returns.
- Throughout the above argument we did not distinguish the risky assets and riskless assets, i.e., the portfolio rebalancing could be implemented for both kinds of assets.
- For a portfolio with two risky assets, from **(2.12)** we can write:

$$\underbrace{W_{t+1} - W_t}_{\text{The change in the value invested in the portfolio}} = \sum_{d=0}^{1} \vartheta_t^i . (S_{t+1}^i - S_t^i) = \underbrace{\vartheta_t^1 . (S_{t+1}^1 - S_t^1)}_{\text{The value change in risky asset 1}} + \underbrace{\vartheta_t^2 . (S_{t+1}^2 - S_t^2)}_{\text{The value change in risky asset 2}}$$

$$W_t = W_0 + \sum_{i=0}^{t-1} \vartheta_i^1 . (S_{i+1}^1 - S_i^1) + \sum_{i=0}^{t-1} \vartheta_i^2 . (S_{i+1}^2 - S_i^2)$$

for $t = 1,\dots,T$

$$W_0 = \vartheta_0^2 . S_0^2 + \vartheta_0^1 . S_0^1 \quad \text{at } t = 0$$

2.3. Discounting and discounted assets

We should give out necessary and sufficient conditions where a finance market $\bar{S}$ satisfies **NA** as in **Definition 2.1** above. Then we introduce two further notations based on which the basic concepts of finance markets (like no-arbitrage, **NA)** should not and do not depend on the choice of any unit for pricing assets, i.e., by USD, EURO, etc. For this reason, the first concept of the notion of discounting is introduced on which we are free to change unit so that it makes mathematics simpler. It turns out that in particular a good choice is a unit which itself is a traded asset and the canonical choice is to use the risk-free asset S^0. Then the prices of risky assets are discounted with S^0, and we define the discounted assets $X^0, X^1, \ldots, X^d$ by

$$X_t^i = \frac{S_t^i}{S_t^0}, \ t \in \{0, \ldots, T\}, \ i \in \{1, \ldots, \boldsymbol{d}\} \ (2.14)$$

Then $X_t^0 \equiv 1$ and $X_t = \{X_t^1, \ldots, X_t^d\}$ express the values of the risky assets at time t in comparing with units of the risk-free asset as a numérais S_t^0.

The mathematic reason for taking a trade asset as a numérais is that it allows to reduce the dimension of the market from $1 + \boldsymbol{d}$ to $\boldsymbol{d}$, and in this case the values of the risky assets are estimated with the value of the risky-free asset. This process is called discounting.

Example 2.4. Consider again a binominal model with one period in the example 1. The discounted risky asset X^1 given by $X_0^1 = 1$

$$X_1^1(\omega_1) = \frac{1+u}{1+r} \quad and \quad X_1^1(\omega_2) = \frac{1+d}{1+r}$$

Now we can reformulate the notion of arbitrage in terms of the discounted risky assets $X = \{X^1, \ldots, X^d\}$ only.

Proposition 1. The following statements are equivalent

a) The market $\bar{S}$ satisfies **NA**

b) The discounted risky assets X satisfy **NA** i.e., there does not exist an arbitrage opportunity $\vartheta = (\vartheta^1, \ldots, \vartheta^d) \in \mathbf{R}^d$ for X such that

$$\vartheta.(X_1 - X_0) \geq 0 \quad P_{-a.s.}$$

$$\text{and} \quad P(\vartheta.(X_1 - X_0) > 0) > 0$$

Proof

We should start to proof the direction from a) to b) which is also more difficult direction. And the statement a) asserts that there is not existence of an arbitrage opportunity $\bar{\vartheta} := (\vartheta^0, \vartheta) \in \mathbf{R}^{d+1}$ for the finance market $\bar{S}$ such that $\bar{\vartheta}.\bar{S}_0 \leq 0$ $P_{-a.s.}$, and $\bar{\vartheta}.\bar{S}_1 \geq 0$ with $P(\bar{\vartheta}.\bar{S}_1 > 0) > 0$ (by **Definition 2.1**). Seeking a contradiction, suppose that there exists an arbitrage opportunity $\bar{\vartheta} = \{\vartheta^0, \vartheta^1, \vartheta,^2 \ldots, \vartheta^d\} \in \mathbf{R}^{d+1}$ with $\bar{S} = \{S^0, S^1, S,^2 \ldots, S^d\}$ and the corresponding discounted assets $\bar{X} = \{X^0, X^1, X,^2 \ldots, X^d\}$, i.e., it is satisfies **Definition 2.1**. Then we obtain

$$\bar{\vartheta}.\bar{S}_0 \leq 0, P_{-a.s.} \text{ and } \bar{\vartheta}.\bar{S}_1 \geq 0 \text{ with } P(\bar{\vartheta}.\bar{S}_1 > 0) > 0$$

and write in more details

$$\vartheta^0.1 + \vartheta.S_0 \leq 0, P_{-a.s.}; \quad \vartheta^0.S_1^0 + \vartheta.S_1 \geq 0 \text{ with } P(\vartheta^0.S_1^0 + \vartheta.S_1 > 0) > 0$$

$$\vartheta^0 + \vartheta.\frac{S_0}{S_0^0} \leq 0, P_{-a.s.}; \quad \vartheta^0.\frac{S_1^0}{S_1^0} + \vartheta.\frac{S_1}{S_1^0} \geq 0 \text{ with } P\left(\vartheta^0.\frac{S_1^0}{S_1^0} + \vartheta.\frac{S_1}{S_1^0} > 0\right) > 0$$

$$\vartheta^0 + \vartheta.X_0 \leq 0, P_{-a.s.}; \quad \vartheta^0 + \vartheta.X_1 \geq 0 \text{ with } P(\vartheta^0 + \vartheta.X_1 > 0) > 0$$

Taking the difference of $\vartheta^0 + \vartheta.X_1 \geq 0$ and $\vartheta^0 + \vartheta.X_0 \leq 0$, $P_{-a.s.}$ gives

$$\vartheta(X_1 - X_0) \geq 0, P_{-a.s.},$$

and of course $P(\vartheta(X_1 - X_0) > 0) > 0$ (2.15)

Now we aim to extend the trade opportunity ϑ into an arbitrage opportunity $\bar{\vartheta}$ on the market $\bar{S}$ by choosing ϑ^0 in an appropriate way, setting

$$\vartheta^0 := -\vartheta.X_0$$

Then with $\bar{\vartheta} := (\vartheta^0, \vartheta)$ and $\bar{X}_0 := (X_0^0, X_0) = (1, X_0)$, we have

$$\bar{\vartheta}.\bar{X}_0 = \vartheta^0.X_0^0 + \vartheta.X_0 = \vartheta^0 + \vartheta.X_0$$

$$= -\vartheta.X_0 + \vartheta.X_0 = 0$$

Multiplying the above expression by $S_0^0 = 1$ gives

$$(\bar{\vartheta}.\bar{X}_0).S_0^0 = (\vartheta^0.X_0^0 + \vartheta.X_0)S_0^0$$

$$= \vartheta^0.S_0^0 + \vartheta.S_0 = \bar{\vartheta}.\bar{S}_0 = 0 \ (2.16)$$

And next adding $X_1^0 - X_0^0 = 1 - 1 = 0$ into the expression (**2.15**) gives

$$\vartheta^0.(X_1^0 - X_0^0) + \vartheta.(X_1 - X_0)$$

$$= \bar{\vartheta}.(\bar{X}_1 - \bar{X}_0) \geq 0 \quad P_{-a.s.}$$

$$\text{And} \quad P(\bar{\vartheta}.(\bar{X}_1 - \bar{X}_0) > 0) > 0 \ (2.17)$$

The expression (**2.15**) and (**2.17**) show that $\bar{\vartheta}.(\bar{X}_1 - \bar{X}_0) > 0$ and $\vartheta.(X_1 - X_0) > 0$ have the same value (due to setting $\vartheta^0 := -\vartheta.X_0$), and also show that $\bar{\vartheta}.(\bar{X}_1 - \bar{X}_0) > 0$ is performed by co-occurrence of $\vartheta^0.(X_1^0 - X_0^0) = 0$ with $P = 1$ and $\vartheta.(X_1 - X_0) > 0$ with $P > 0$.

Plugging (**2.16**) into (**2.17**) gives

$$\bar{\vartheta}.\bar{X}_1 \geq 0 \quad P_{-a.s.} \quad \text{and} \quad P(\bar{\vartheta}.\bar{X}_1 > 0) > 0$$

By using that the inequalities remain unchanged by multiplying it with a positive constant, here S_1^0, we have

$$\bar{\vartheta}.\bar{S}_1 \geq 0 \quad P_{-a.s.} \quad \text{and} \quad P(\bar{\vartheta}.\bar{S}_1 > 0) > 0$$

This in combination with (**2.16**) shows that $\bar{\vartheta}$ is an arbitrage opportunity for the finance market $\bar{S}$, in contradiction to the hypothesis of the part a) in **proposition 1** that $\bar{S}$ satisfies **NA**.

In the next, to prove the direction from b) to a) it is clearly that by the part b) in **proposition 1**, there is not exists a trading portfolio $\vartheta = \{\vartheta^1, \vartheta^2, . . . , \vartheta^d\} \in \mathbf{R}^d$ such that

$$\vartheta.(X_1 - X_0) \geq 0 \quad P_{-a.s.}$$
$$\text{and} \quad P(\vartheta.(X_1 - X_0) > 0) > 0$$

This implies that the trading portfolio satisfies the complement of $\vartheta.(X_1 - X_0) \geq 0$, i.e.,

$$\vartheta.(X_1 - X_0) < 0 \quad P_{-a.s.}$$
$$\text{And} \quad P(\vartheta.(X_1 - X_0) \leq 0) > 0 \quad (2.18)$$

Repeat the steps in the proof for the direction from a) to b) but in reverse, and set

$$\vartheta^0 := -\vartheta.X_0$$

We obtain

$$\bar{\vartheta}.\bar{X}_0 = \vartheta^0.X_0^0 + \vartheta.X_0 = \vartheta^0 + \vartheta.X_0 = -\vartheta.X_0 + \vartheta.X_0 = 0 \quad (2.19)$$

This together with (**2.18**) show that

$$\vartheta.X_1 - \vartheta.X_0 < 0 \quad P_{-a.s.}$$
$$\text{and} \quad P(\vartheta.X_1 - \vartheta.X_0 \leq 0) > 0$$
$$\vartheta.X_1 + \vartheta^0 < 0 \quad P_{-a.s.}$$
$$\text{and} \quad P(\vartheta.X_1 + \vartheta^0 \leq 0) > 0$$

Recall the definition of the discounted asset

$$X_i^0 = \frac{S_i^0}{S_i^0} = 1 \quad \text{for} \quad i \in \{0,1\} \Rightarrow X_0^0 = X_1^0 = 1$$

And write

$$\vartheta.X_1 + \vartheta^0.X_1^0 < 0 \quad P_{-a.s.}$$
$$\text{and} \quad P((\vartheta.X_1 + \vartheta^0.X_1^0) \leq 0) > 0$$
$$\bar{\vartheta}.\bar{X}_1 < 0 \quad P_{-a.s.} \quad \text{and} \quad P((\bar{\vartheta}.\bar{X}_1) \leq 0) > 0$$

Multiplying by a positive constant S_1^0 gives

$$\bar{\vartheta}.\bar{S}_1 < 0 \quad P_{-a.s.} \text{ and } P(\bar{\vartheta}.\bar{S}_1 \leq 0) > 0 \quad (2.20)$$

The expression (**2.20**) indicates that $\bar{\vartheta}.\bar{S}_1 < 0 = \bar{\vartheta}.\bar{S}_0$ $P_{-a.s.}$ does not respond to the **Definition 2.1** for an arbitrage portfolio, i.e., the hypothesis of the part b) leads to that there is not exist the portfolio for the finance market $\bar{S}$. In other words, it satisfies **NA**.

2.4. The fundamental theorem of asset pricing (FTAP)

Before presenting **FTAP** we need to give the definition of concept of an equivalent martingale measure (**EMM**).

Definition 2.3. A discrete-time stochastic process $\{M_t\}_{t=0}^T$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a martingale with respect to X if

a) M_t is integrable for all $t = 0, \ldots, T$, and $\boldsymbol{E}[|M_t|] < \infty$.

b) The conditional expectation of M_t given $X_s, X_{s-1}, \ldots, X_0$ is equal to M_s,

$$\boldsymbol{E}[M_t | X_s, X_{s-1}, \ldots, X_0] = M_s \quad \text{for} \quad 0 \leq s < t.$$

Definition 2.4. Let $(\Omega, \mathcal{F})$ be a measurable space. Two probability measures P and Q on $(\Omega, \mathcal{F})$ are called equivalent (notation: $P \approx Q$) if for $A \in \mathcal{F}$, $Q(A) = 0$ if and only if $P(A) = 0$.

Now we can define the concept of an equivalent martingale measure **EMM**.

Definition 2.5. Let X be the discounted risky assets on the probability space $(\Omega, \mathcal{F}, P)$. A measure Q on $(\Omega, \mathcal{F})$ is called an **equivalent martingale measure** for X if $Q \approx P$, each X^i is Q −integrable and

$$\boldsymbol{E}^Q[X_1^i] = X_0^i \quad \text{for} \quad i = (1, \ldots, \boldsymbol{d})$$

Here the terminology equivalent martingale measure stems from the fact that the $X^{i\prime}s$ are martingales under the equivalent measure Q.

In the following we consider the fundamental theorem of asset pricing for the one-period finance market. For the multiperiod discrete-time finance markets [5], we refer the readers to some articles such as "Arbitrage and duality in nondominated discrete-time models, by Bruno Bouchard and Marcel Nutz, The annals of applied probability, 2015, Vol. 25, No.2, pages 823 - 859".

Theorem 1. The fundamental theorem of asset pricing (**FTAP**)

Let $\bar{S} = (S_t^0, S_t), t \in \{0,1\}$ be one-period finance market on some probability $(\Omega, \mathcal{F}, P)$. The following statements are equivalent

a) The market satisfies **NA**.

b) There exists an **EMM** for the discounted risky assets $X = S/S^0$.

Proof

Normally we will start in the easy direction from a) to b). And let $P \approx Q$ be an EMM. So, by using **Definition 2.5** we have $\boldsymbol{E}^Q[X_1^i] = X_0^i$ for $i = (1, \dots, \boldsymbol{d})$ then $\boldsymbol{E}^Q[\vartheta.(X_1 - X_0)] = 0\ Q_{-a.s.}$ and $Q(\vartheta.(X_1 - X_0) > 0) = 0$. By **Proposition 1**, It suffices to show that X satisfies **NA**, i.e., that there not exists a trade portfolio $\vartheta = \{\vartheta^1, \vartheta,^2 \ . \ . \ . \ , \vartheta^d\} \in \mathbf{R}^d$ for X such that

$$\vartheta.(X_1 - X_0) \geq 0 \quad Q_{-a.s.}$$

$$\text{and } Q(\vartheta.(X_1 - X_0) > 0) > 0$$

To prove theorem FTAP by seeking a contradiction, we assume that there exists a financial market with $\vartheta = \{\vartheta^1, \vartheta,^2 \ . \ . \ . \ , \vartheta^d\} \in \mathbf{R}^d$ such that

$$\vartheta.(X_1 - X_0) \geq 0 \quad P_{-a.s.}$$

$$\text{and } \quad P(\vartheta.(X_1 - X_0) > 0) > 0$$

By the fact that Q is equivalent to P we can write

$$\vartheta.(X_1 - X_0) \geq 0 \quad Q_{-a.s.}$$

$$\text{and } \quad Q(\vartheta.(X_1 - X_0) > 0) > 0$$

By the monotonicity of the expectation operator, we have

$$\boldsymbol{E}^Q[\vartheta.(X_1 - X_0)] > 0$$

But by linearity of the expectation operator, we can write $\boldsymbol{E}^Q[\vartheta.(X_1 - X_0)]$ in more details then using the fact that Q is **EMM** for X

$$\boldsymbol{E}^Q[\vartheta.(X_1 - X_0)] = \boldsymbol{E}^Q\left[\sum_{i=1}^{d} \vartheta^i (X_1^i - X_0^i)\right]$$

$$= \sum_{i=1}^{d} \vartheta^i \boldsymbol{E}^Q[(X_1^i - X_0^i)] = \sum_{i=1}^{d} \vartheta^i \text{x } 0 = 0$$

And $\vartheta.(X_1 - X_0) = 0$ is in a contradiction to $\vartheta.(X_1 - X_0) > 0$.

Thus, this contradiction shows that there not exists a trading portfolio $\vartheta = (\vartheta^1, \vartheta,^2 \ . \ . \ . \ , \vartheta^d) \in \mathbf{R}^\mathbf{d}$ for S market, and therefor for $\bar{S}$ market. We conclude that the existence of an EMM for finance markets does not allow the hypothesis

$$\vartheta.(X_1 - X_0) \geq 0 \quad P_{-a.s.}$$

$$\text{and } \quad P(\vartheta.(X_1 - X_0) > 0) > 0$$

meaning that the finance market satisfies **NA**.

Now we can prove the reverse direction from a) to b). this direction stems from the hypothesis of the existence of the $\bar{S}$ market satisfying **NA**, i.e., there exists conditional expected value of the portfolio where conditional probability for each state of every risky asset is defined. Here the existence of an **EMM** for discounted risky assets must be proven. This requires the world of states where there exists $P \approx Q$ and the conditional expected value of each discounted risky asset is evaluated.

For the proof, we only consider the special case that $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$ is finite and containing all outcome states of all risky assets, $\mathcal{F} = 2^\Omega$ and $P(\omega_i) > 0$ for all $i \in \{1, \dots, N\}$. But in the case Ω is infinite countable we can easily prove by induction [6]. For the general case we also refer the readers an excellent book, H. Föllmer and A. Schied, *Stochastic Finance*, 4th ext. ed. De Gruyter Studies in Mathematics, vol. 27, Walter de Gruyter & Co., Berlin 2016.

We first consider a random variable Y on some measurable space $(\Omega, \mathcal{F})$ with the $\mathbf{R}^d$-valued vector $\big(Y(\omega_1), Y(\omega_2), \dots, Y(\omega_N)\big)$. Based on that the market $\bar{S}$ satisfies **NA**, i.e., the value $\vartheta.(X_1 - X_0) \leq 0$ is created after each period. Then we set

$$\mathcal{K} := \{\vartheta.(X_1(\omega_1) - X_0), \vartheta.(X_1(\omega_2) - X_0), \dots, \vartheta.(X_1(\omega_N) - X_0) \colon \vartheta \in \mathbf{R}^\mathbf{d}\} \qquad (2.21)$$

A such $\mathcal{K}$ corresponds to the collection of all random variables in the form $\vartheta.(X_1 - X_0)$ for $\vartheta \in \mathbf{R}^d$. Note here that each $\mathcal{K}$-element $\vartheta.(X_1(\omega_i) - X_0)$ is the portfolio return created when the portfolio is evaluated at states ω_i, it can take a value negative, positive or zero. Mathematically, a subspace $\mathcal{K}$ (minimum $\boldsymbol{d}$ −dimensions, maximum N −dimensions) is a vector space of $\mathbf{R}^N$ with all elements that satisfies **NA** by **proposition 1**, that is $\vartheta.(X_1(\omega_i) - X_0) \le 0$ for $i \in \{1, ..., N\}$. So, we have

$$\mathcal{K} \cap \mathbf{R}_+^N = \{\mathbf{0}\}$$

where $\mathbf{R}_+^N = [0, +\infty)^N$, a space of the N −dimensions vectors with their each component that is a positive number; $\{\mathbf{0}\}$: zero vector.

To set up some probability measure Q on $(\Omega, \mathcal{F})$, we consider a subspace $\Delta^{N-1} \subset \mathbf{R}_+^N$, and $\mathbf{0} \notin \Delta^{N-1}$, and define the standard simplex of dimension $N-1$

$$\Delta^{N-1} := \left\{ \boldsymbol{x} \in \mathbf{R}_+^N : \sum_{i=1}^{N} x^i = 1 \right\} \qquad (2.22)$$

where the index i here is not exponent, but only the i^{th} −component with its value $x^i > 0$; $\boldsymbol{x} \in \mathbf{R}_+^N$ is a vector with N components, $0 < x^i < 1$ for $i \in \{1, ..., N\}$, which satisfies $\sum_{i=1}^{N} x^i = 1$. It means that $\Delta^{N-1} \subset \mathbf{R}_+^N$ and $\{\mathbf{0}\} \notin \Delta^{N-1}$, so

$$\mathcal{K} \cap \Delta^{N-1} = \emptyset$$

As $\mathcal{K}$ and Δ^{N-1} are both nonempty and convex, $\mathcal{K}$ is closed, and Δ^{N-1} is compact, it follows from the strict separating hyperplane theorem (see D. Bertsekas, *Nonlinear Programming*, second Ed. Athena Scientific, Belmont, MA, 1999 [7]). As an illustration we consider the case with $N = 2$, then every vector $\boldsymbol{x} \in \Delta^{N-1}$ possesses two components, x^1 and x^2 with their supremum and infimum that are of 1 and zero, respectively. It follows from that each Δ^{N-1} −component x^i is positioned at a point located in a range of (0,1) as illustrated in the following figure.

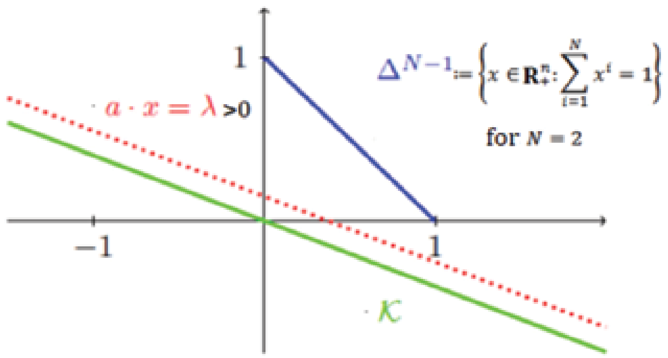


Figure 2.3. Illustration of the separating hyperplane theorem for $N = 2$, [8].

And there is the fact that $\mathcal{K} \subset \mathbf{R}^N$ is a vector subspace that there exists a vector $\boldsymbol{a} \in \mathbf{R}^N \backslash \{\mathbf{0}\}$ perpendicular to a vector $\boldsymbol{k} \in \mathcal{K}$, i.e.,

$$\boldsymbol{a}.\boldsymbol{k} = 0 \quad \text{for all } \boldsymbol{k} \in \mathcal{K}$$

Also there exists a real number $\lambda > 0$ such that

$$\boldsymbol{a}.\boldsymbol{x} \ge \lambda > 0 \quad \text{for all} \quad \boldsymbol{x} \in \Delta^{N-1}$$

As Δ^{N-1} contains all standard unit vectors e^i in $\mathbf{R}^N$. It follows that

$$\boldsymbol{a}.e^i = \boldsymbol{a}^i > 0 \quad \text{for } i \in \{1, ..., N\}$$

Now we define the probability measure Q on $(\Omega, \mathcal{F})$ by

$$Q(\omega_i) = \frac{a_i}{\sum_{k=1}^{N} a_k} > 0$$

where $a_i > 0$ is the i^{th} −component of the vector $\boldsymbol{a}$, and of course $\sum_{i=1}^{N} Q(\omega_i) = 1$. It follows from **Definition 2.4** that $Q \approx P$ on $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$.

To the end we need to define the conditional expectation of the risky assets on Q. And we consider vector $\boldsymbol{k}^i \in \mathcal{K}$ as follows

$$\boldsymbol{k}^i = \left(e^i.(X_1(\omega_1) - X_0), \dots, e^i.(X_1(\omega_N) - X_0)\right) \in \mathcal{K}, \text{for } i \in \{1, \dots, d\}$$

where e^i denotes the unit vector in $\mathbf{R}^d$.

The expectation of $[X_1^i - X_0^i]$ is estimated on $(\Omega, \mathcal{F}, Q)$

$$E^Q[X_1^i - X_0^i] = \sum_{n=1}^{N} (X_1^i(\omega_n) - X_0^i).Q(\{\omega_n\}) =$$

$$= \frac{1}{\sum_1^N a_k} \sum_{n=1}^{N} a_n.(X_1^i(\omega_n) - X_0^i) =$$

$$= \frac{1}{\sum_1^N a_k} \sum_{n=1}^{N} a_n.\left(e^i.(X_1(\omega_n) - X_0)\right) =$$

$$= \frac{1}{\sum_1^N a_k} \sum_{n=1}^{N} a_n.k_n^i = \frac{a.k^i}{\sum_1^N a_k} = 0$$

In the last step we used the result above $\boldsymbol{a}.\boldsymbol{k} = 0$ for all $\boldsymbol{k} \in \mathcal{K}$. Finally

$$E^Q[X_1^i - X_0^i] = 0 \Rightarrow E^Q[X_1^i] = E^Q[X_0^i] = M_0 \text{ for } i \in \{1,2,\dots,\mathbf{d}\}$$

By **Definition 2.5**, Q is called an equivalent martingale measure on $(\Omega,\mathcal{F})$ for X.

Example 2.5. Consider the binominal model from **Example 2.4**, that describes a finance market with one risky asset with two states "up" and "down" at the end of each period

$$S^1(\omega_1) = 1 + u \text{ and } S^1(\omega_2) = 1 + d$$

By using **FTAP** we want to check when the finance market satisfies **NA**. So let Q be a measure on $(\Omega,\mathcal{F})$, and set $q_1 \coloneqq Q(\omega_1)$ and $q_2 \coloneqq Q(\omega_2)$. By **Definition 2.4**, $Q \approx P$ if and only if $q_1 > 0$ and $q_2 > 0$. Indeed, due to $p_1 \coloneqq P(\omega_1) > 0$ and $p_2 \coloneqq P(\omega_2) > 0$ then $q_1 > 0$ and $q_2 > 0$ because of that if $q_1 = 0$ or $q_2 = 0$ or both then the equivalence of relations on two probability measures leads to $p_1 = 0$ or $p_2 = 0$ or both $p_1 = 0$ and $p_2 = 0$. It is a contradiction. Moreover Q is an **EMM** if and only if (by **Definition 2.5**)

$$E^Q[X_1^1] = X_0^1 = 1$$

$$\Leftrightarrow q_1 X_1^1(\omega_1) + q_2 X_1^1(\omega_2) = 1$$

$$\Leftrightarrow q_1 \frac{1+u}{1+r} + q_2 \frac{1+d}{1+r} = 1$$

Using $q_2 = (1 - q_1)$ then rearranging gives

$$q_1(1+u) + (1-q_1)(1+d) = 1 + r$$

$$\Leftrightarrow q_1 = \frac{r-d}{u-d}$$

$$q_2 = 1 - q_1 = \frac{u-r}{u-d}$$

Clearly this result shows that $q_1 > 0$ and $q_2 > 0$ if and only if $u > r > d$. So the finance market $\bar{S}$ satisfies **NA** if and only if $u > r > d$, in which case there exists only one **EMM** satisfying

$$q_1 = \frac{r-d}{u-d} \quad and \quad q_2 = \frac{u-r}{u-d}$$

The condition $u > r > d$ is quite intuitive from economics perspective as it says that the risky assets must offer the chance of a higher return than the interest rate in one state of the world, state $u > r$ but also have a lower return than the interest rate in another state $d < r$. Note here that the expression of q_1 and q_2 there not appears p_1 or p_2, i.e., the **EMM** Q does not depend on the value of p_1 and p_2.

3. Risk measures and value-at-risk

The risk management of asset investment is very important for optimization of the portfolio and maximization of the portfolio return. Since 1990's value-at-risk **(VaR)** and conditional value-atrisk **(C-VaR)** have widely used as risk measures, replacing the traditional variance measure. **VaR** is the threshold point of a specific lower percentile on the return distribution, whereas **C-VaR** is the expected loss beyond the **VaR** level in the distribution's lower tail. These risk measures are particularly useful when the investors consider the downside risk of a financial position. Simply **VaR** is the worst loss over the target horizon that will not be exceeded with a given level of confidence. Losses can occur through a combination of two factors: the volatility in the underlying financial variable and the exposure to this source of risk. Corporations have no control over the volatility of financial variables, they can adjust their exposure to these risks, for instance, through derivatives. **VAR** captures the combined effect of underlying volatility and exposure to financial risks. To this end we will present the relevant content at the basic level according to the axiom approach initiated by Artzner, Delbaen, Eber & Heath in a seminar paper (1999), the theory of monetary risk measures [9].

Definition 3.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathfrak{X}$ the set of all random variables. Let $\alpha \in (0,1)$ be a confidence level. For $X \in \mathfrak{X}$, the value at risk of X at level α is given by

$$\mathbf{VaR}_\alpha(X) = \inf\{m \in \mathbf{R}: P\big((X+m) < 0\big) \le \alpha\}$$

The **VaR** at α level is the smallest amount of capital which, if added to X, keeps the probability of a negative outcome below or equal to α. Typical value for α are 0.05, 0.01 or 0.001. **VaR** is probably the most widely used risk measure in practice. One can easily check that it is normalized, monotone, cash-invariant and positively homogeneous. However, Value at Risk fails to be convex, i.e., **VaR** has its limitations instead of encouraging it. Theoretically **VaR** is lacking in sub-additivity and neglects the risk beyond the threshold [9]. Rockafellar and Uryasev showed that VaR is biased just for the optimum portfolio [10].

Example 3.1. Let X^1 and X^2 be two independent-identically distributed random variables ($i.i.d.$ variables) on some probability $(\Omega, \mathcal{F}, P)$, where

$$P\big(X^i = -100\big) = 0.01 \text{ and } P\big(X^i = 90\big) = 0.99 \text{ for } i = \{1,2\}$$

Then for $i = \{1,2\}$

$$X^i + m < 0$$

$$\text{when } \begin{cases} X^i = 90 \text{ then } m \in (-\infty, -90) \\ X^i = -100 \text{ then } m \in [-90,100) \\ \text{it does not exist if } m \in [100, \infty) \end{cases}$$

And for $i = \{1,2\}$ if $X^i = \{-100, 90\}$ then

$$P(X^i + m < 0) = \begin{cases} 1 & \text{if } m \in (-\infty,-90) \text{ then only exists } X^i + m < 0 \\ 0.01 & \text{if } m \in [-90,100) \text{ then } X^i + m = -100 \\ 0 & \text{if } m \in [100,\infty) \text{ then } X^i + m > 0 \end{cases}$$

So the smallest amount of capital which must be added to X to keep the probability of a negative outcome $(X^i + m) < 0$ at level 0.01 is

$$\mathbf{VaR}_{0.01}\big(X^i\big) = -90 \text{ for } i = \{1,2\}$$

Therefore both X^1 and X^2 are acceptable and even very good from a $\mathbf{VaR}_{0.01}\big(X^i\big) = -90$ perspective. Now we consider the "diversified position" for investments in X^1 and X^2. Let

$$X := \frac{1}{2}X^1 + \frac{1}{2}X^2$$

Then the distribution of X satisfies (X could be taken one of three values by $-100, -5, 90$ with the corresponding probability)

$$P(X = -100) = (0.01)^2 = 0.0001$$

(Since both X^1 and X^1 are simultaneously taken value by -100)

$$P(X = -5) = 2 \text{ x } 0.01 \text{ x } 0.99 = 0.0198$$

(Since there exist two choices $X^1 = 90$ and $X^2 = -100$, or $X^1 = -100$ and $X^2 = 90$)

$$P(X = 90) = (0.99)^2 = 0.9801$$

(Since both X^1 and X^2 are simultaneously gotten value by 90)

And we have $P(X + m < 0)$

$$= \begin{cases} 1 & \text{if } m \in (-\infty, -90) \text{ then only exist } X + m < 0 \\ 0.0198 & \text{if } m \in [-90, 5) \text{ then } X + m < 0 \\ 0.0001 & \text{if } m \in [5, 100) \text{ then } X + m < 0 \\ 0 & \text{if } m \in [100, \infty) \text{ then only exist } X + m > 0 \end{cases}$$

As resulted

$$\mathbf{VaR}_{0.0001}(X) = 5 \text{ for } i \in \{1,2\}$$

And hence in this case the "diversified position" X is no longer acceptable.

We should consider what is the main difference of the two finance positions above. As computed the probability of the loss, i.e., for $X < 0$, is $P(X < 0) = 0.0198 + 0.0001 = 0.0199$ higher than the probability for X^i, $P\big(X^i < 0\big) = 0.01$ for $i \in \{1,2\}$.

However, the expected size of the loss when it does happen is much larger for X^i

$$\boldsymbol{E}\big[-X^i \big| -X^i \ge 0\big] = \frac{-(-100) \text{ x } 0.01}{0.01} = 100 \text{ for } i \in \{1,2\}$$

than for X

$$\boldsymbol{E}[-X | -X \ge 0] = \frac{-(-100)\text{x } 0.0001 + [-(-5)\text{x } 0.0198]}{0.0001 + 0.0198} = 5.477$$

The results estimated above indicate that even though the probability of the loss is a bit higher for X than for X^i, i.e., $P(X < 0) = 0.0199 > P\big(X^i < 0\big) = 0.01$, but the expected loss for X, equal to 5.477, given default is significantly lower than the expected loss for X^i, equal to 100. Therefore, from a regulatory point of view, X is a much better risk than X^i.

For this reason, investors might look at a more conservative way, i.e., must determine the extent and probabilities of potential losses in the portfolios given default. In this example we have seen that **VaR** at some fixed levels α is positively homogeneous but not convex on the full space $(\Omega, \mathcal{F}, P)$, because of that we can take two events A_i $(i = 1,2)$ such that $P(A_i) \le \alpha$, but $P(A_1 \cap A_2) > \alpha$ and thus it is not a coherent risk measure. However, we can take its average then we obtain the basic example of a coherent risk measure, known as Average Value at Risk, or Conditional Value at Risk, or Expected Shortfall, and or Tail Value at Risk [11].

Definition 3.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathfrak{X}$ the set of all real-valued random variables having finite first moments. Let $\alpha \in (0,1)$ be a confidence level. For $X \in \mathfrak{X}$, the expected shortfall of X at level α is given by

$$\mathbf{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathbf{VaR}_u\,(X)du$$

Since $\mathbf{VaR}_u$ is nonincreasing in u [10], it follows that $\mathbf{ES}_\alpha(X) \ge \mathbf{VaR}_u(X)$ for all α and all X on the full space $(\Omega, \mathcal{F}, P)$. It means that $\mathbf{ES}_\alpha(X)$ prescribes higher capital requirements than $\mathbf{VaR}_u(X)$. In fact, $\mathbf{ES}_\alpha(X)$ can be characterized as the smallest distribution-based convex risk measure that dominates $\mathbf{VaR}_u(X)$. Moreover, one can show that unlike Value at Risk, expected Shortfall is a coherent risk measure, i.e., it is a convex risk measure. One can even show that it is "optimal" in the sense that it is the smallest law-invariant convex risk measure that is more conservative than Value at Risk, see [6-Theorem 4.67, 12].

To continue we state an alternative characterization of Expected Shortfall for continuous distributions, which shows that Expected Shortfall takes care of the size of the loss given default.

Lemma 1. Let X be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that the distribution of X is continuous. Then for $\alpha \in (0,1)$

$$\mathbf{ES}_\alpha(X) = \boldsymbol{E}[-X|-X \ge \mathbf{VaR}_\alpha(X)]$$

Note here that Expected Shortfall in lemma 3.1 is defined under continuous distribution of X and encourages diversification and it will false without the assumption of a continuous distribution as seen in the following example.

Example 3.2. Consider the setup of **Example 3.1**, we recall

$$P(X^i + m < 0) = \begin{cases} 1 \text{ if } m \in (-\infty, -90) \text{ then only exists } X^i + m < 0 \\ 0.01 \quad \text{if } m \in [-90, 100) \text{ then } X^i + m = -100 \\ 0 \quad \text{if } m \in [100, \infty) \text{ then } X^i + m > 0 \end{cases}$$

and **Definition 3.1**

$$\mathbf{VaR}_\alpha(X) = \inf\{m \in \mathbf{R}: P\big((X+m) < 0\big) \le \alpha\}$$

Then we obtain for $i \in \{1,2\}$

$$\mathbf{VaR}_u(X^i) = \begin{cases} 100 \text{ if } u = (0, 0.01) \\ -90 \text{ if } u = [0.01, 1) \end{cases}$$

For $i \in \{1,2\}$ this gives

$$\mathbf{ES}_{0.01}(X^i) = \frac{1}{\alpha}\int_0^\alpha \mathbf{VaR}_u\,(X)du = \frac{1}{0.01}\int_0^{0.01} 100\,du = 100$$

Beside this, **Example 3.1** also gives

$$P(X + m < 0) = \begin{cases} 1 \text{ if } m \in (-\infty, -90) \text{ then only exist } X + m < 0 \\ 0.0198 \quad \text{if } m \in [-90, 5) \text{ then } X + m < 0 \\ 0.0001 \quad \text{if } m \in [5, 100) \text{ then } X + m < 0 \\ 0 \quad \text{if } m \in [100, \infty) \text{ then only exist } X + m > 0 \end{cases}$$

Then we obtain for X

$$\mathbf{VaR}_u(X) = \begin{cases} 100 \text{ if } u = (0, 0.0001) \\ 5 \text{ if } u = (0.0001, 0.0199) \\ -90 \text{ if } u = (0.0199, 1) \end{cases}$$

This gives

$$\mathbf{ES}_{\alpha=0.01}(X) = \frac{1}{\alpha}\int_0^\alpha \mathbf{VaR}_u\,(X)du = \frac{1}{0.01}\int_0^{0.0001} 100\,du + \frac{1}{0.01}\int_{0.0001}^{0.01} 5\,du = \frac{1}{0.01}(100 \text{ x } 0.0001 + 5 \text{ x } 0.0099) = 5.95$$

So the $\mathbf{ES}_{\alpha=0.01}(X) = 5.95$ of the "diversified position" X is significantly lower than the Expected Shortfall of the individual position X^i, $\mathbf{ES}_{0.01}(X^i) = 100$ for $i \in \{1,2\}$.

Now we can check **Lemma 1** for this case. We can compute the right side of the expression of the **Lemma 1**

$$\boldsymbol{E}[-X|-X \ge \mathbf{VaR}_\alpha(X)] = \boldsymbol{E}[-X|-X \ge \mathbf{VaR}_{0.01}(X) = 5]$$

$$= \frac{100 \times 0.0001 + 5 \times 0.0198}{0.0199} \simeq 5.47$$

and to compare with the left side of the expression of the **Lemma 1**, $\mathbf{ES}_{\alpha=0.01}(X) = 5.95$. It gives $E[-X|-X \geq \mathbf{VaR}_{0.01}(X) = 5] \simeq 5.47 < \mathbf{ES}_{\alpha=0.01}(X) = 5.95$ and shows that **Lemma 1** is not true in this case where the continuity of the distribution is faulty.

Conclusion

The goal of this paper is to build a collection of fundamental concepts for more insights at basic level for finance market based on which one can establish and determiner preferable stock or financial asset selections and to include the portfolio selection problems related to Value-at-Risk and Expected shortfall. The procedure could be summarized as follows. Consider the price and the return of every stock or each financial asset in a time-series S_t^i and R_t^i for each selected investment period, by using a binominal model. Then compute the $\mathbf{VaR}_\alpha(X^i)$ and $\mathbf{ES}_\alpha(X^i)$ for each stock (X^i) for a period and include for the selected portfolios. It follows from that the investors will turn out the portfolio with their criteria.

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